On The Cohomological Dimension of Local Cohomology Modules

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Dedicated to the memory of Alexander Grothendieck

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Abstract

Let R be a Noetherian ring, I an ideal of R and M an R-module with $\operatorname{cd}(I, M) = c$. In this article, we first show that there exists a descending chain of ideals $I = I_c \supsetneq I_{c-1} \supsetneq \cdots \supsetneq I_0$ of R such that for each $0 \leq i \leq c-1$, $\operatorname{cd}(I_i, M) = i$ and that the top local cohomology module $\mathrm{H}^i_{I_i}(M)$ is not Artinian. We then give sufficient conditions for a non-negative integer t to be a lower bound for $\operatorname{cd}(I, M)$ and use this to conclude that in non-catenary Noetherian local integral domains, there exist prime ideals that are not set theoretic complete intersection. Finally, we set conditions which determine whether or not a top local cohomology module is Artinian.

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1 Introduction

Throughout, R denote a commutative Noetherian ring with unity, I an ideal of R. For an R-module M, the i-th local cohomology module of M with support in I is defined as

$$\mathrm{H}^i_I(M) = \varinjlim \operatorname{Ext}^i_R(R/I^n, M).$$

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For details about the local cohomology modules, we refer the reader to [4] and [7].
One of the important invariant related to local cohomology modules is *the cohomological dimension of M with respect to I*, denoted by $\mathrm{cd}(I, M)$, and defined as:

$$\mathrm{cd}(I, M) = \sup\ \{i \in \mathbb{N} \mid\ \mathrm{H}_I^i(M) \neq 0\}.$$

If $M = R$, we write $cd(I)$ instead of $cd(I, R)$.
There are two interesting questions related to local cohomology modules, the first one is to determine the lower and upper bounds for $\mathrm{cd}(I, M)$ and the second one is to determine whether or not $\mathrm{H}_I^i(M)$ is Artinian(see e.g. [1], [5], [6], [8], [9] and [10]).
Our results in this regard are as follows:
In section 2, we show that for an R-module M with $\mathrm{cd}(I, M) = c$, there is a descending chain of ideals

$$I = I_c \supsetneq I_{c-1} \supsetneq \cdots \supsetneq I_0$$

of R such that for each $0 \leq i \leq c$, $\mathrm{cd}(I_i, M) = i$, and that the top local cohomology module $\mathrm{H}_{I_i}^i(M)$ is not Artinian.
In section 3, we prove a result that gives a sufficient condition for an integer to be a lower bound for the cohomological dimension, $\mathrm{cd}(I, M)$, of M at I. One of the important conclusion of this result is that over a Noetherian local ring $(R, \mathfrak{m})$, for a finitely generated R-module M of dimension n and an ideal I of R with $\dim(M/IM) = d \geq 1$, $n - d$ is a lower bound for $\mathrm{cd}(I, M)$ and if moreover $\mathrm{cd}(I, M) = n - d$, then $\mathrm{H}_{\mathfrak{m}}^d(\mathrm{H}_I^{n-d}(M)) \cong \mathrm{H}_{\mathfrak{m}}^n(M)$. As an application of this result, we show that in non-catenary Noetherian local integral domains, there exist prime ideals that are not set theoretic complete intersection.
In section 4, we examine the Artinianness and non-Artinianness of top local cohomology modules.

2 Descending Chains With Successive Cohomological Dimensions

In this section, we prove the existence of descending chains of ideals and locally closed sets with successive cohomological dimensions. The main result of this section is the following:

Theorem 2.1. *Let R be a Noetherian ring, I an ideal of R and M an R-module with $\operatorname{cd}(I, M) = c > 0$. Then there is a descending chain of ideals*

$$I = I_c \supsetneqq I_{c-1} \supsetneqq \cdots \supsetneqq I_0$$

such that $\operatorname{cd}(I_i, M) = i$ for all i, $0 \le i \le c$. Moreover $\mathrm{H}^i_{I_i}(M)$ is not Artinian for all i, $0 \le i \le c-1$.

Proof. Consider the set

$$\mathbb{S} = \{ \ J \subsetneqq I \mid \operatorname{cd}(J, M) < c \ \}.$$

Clearly, the zero ideal belongs to $\mathbb{S}$ and so $\mathbb{S}$ is a non-empty subset of ideals of R. Since R is Noetherian, $\mathbb{S}$ has a maximal element, say I_{c-1}. We claim that $\operatorname{cd}(I_{c-1}, M) = c-1$. To prove this, let $x \in I \setminus I_{c-1}$ and so $I_{c-1} + Rx \subseteq I$. But then it follows from the maximality of I_{c-1} in $\mathbb{S}$ and Remark 8.1.3 of [4] that

$$c \le \operatorname{cd}(I_{c-1} + Rx, M) \le \operatorname{cd}(I_{c-1}, M) + 1 < c + 1.$$

Hence $\operatorname{cd}(I_{c-1} + Rx, M) = c$. Now consider the exact sequence

$$\cdots \longrightarrow \left(\mathrm{H}^{c-1}_{I_{c-1}}(M)\right)_x \longrightarrow \mathrm{H}^c_{I_{c-1}+Rx}(M) \longrightarrow \mathrm{H}^c_{I_{c-1}}(M) = 0 \ .$$

Since $\mathrm{H}^c_{I_{c-1}+Rx}(M)$ is nonzero, it follows that $\left(\mathrm{H}^{c-1}_{I_{c-1}}(M)\right)_x$ is nonzero, then so is $\mathrm{H}^{c-1}_{I_{c-1}}(M)$. Therefore the claim follows.
Iterating this argument, one can obtain a descending chain of ideals, as desired.
For the second part, let $x \in I_{i+1} \setminus I_i$ and consider the ideal $I_i + Rx$. Then it follows from the construction of the ideal I_i that $\operatorname{cd}(I_i + Rx, M) = i + 1$. Now $\mathrm{H}^i_{I_i}(M)$ is non-Artinian follows from Corollary 4.1 of [5]. □

Recall that a subspace Z of a topological space X is said to be *locally closed*, if it is the intersection of an open and a closed set. Let X be a topological space, $Z \subseteq X$ be a locally closed subset of X and let F be an abelian sheaf on X. Then the i^{th} local cohomology group of F with support in Z is denoted by $H^i_Z(X, F)$. We refer the reader to [7] and [11] for its definition and details.
If, in particular, $X = Spec(R)$ is an affine scheme, where R is a commutative Noetherian ring, and $F = M^{\sim}$ is the quasi coherent sheaf on X associated to an R-module M, we write $H^i_Z(M)$ instead of $H^i_Z(X, M^{\sim})$.
The following corollary may be considered as an easy application of our result above.

Corollary 2.2. *Let R be a Noetherian ring, M an R-module and I an ideal of R such that $cd(I, M) = c > 1$. Then there is a descending chain of locally closed sets*

$$T_{c-1} \supsetneq T_{c-2} \supsetneq \cdots \supsetneq T_1$$

in Spec(R) such that $cd(T_i, M) = i$ for all $1 \leq i \leq c-1$.

Proof. Let I be an ideal of R with $cd(I, M) = c > 1$. Then it follows from Theorem 2.1 that there is a descending chain of ideals

$$I = I_c \supsetneq I_{c-1} \supsetneq \cdots \supsetneq I_1 \supsetneq I_0$$

such that $cd(I_i, M) = i$ for all $0 \leq i \leq c$. Let now $U_i = V(I_i)$ and define the locally closed sets $T_i := U_1 \setminus U_{i+1}$. Then it is easy to see that

$$T_{c-1} \supsetneq T_{c-2} \supsetneq \cdots \supsetneq T_1.$$

On the other hand, it follows from Proposition 1.2 of [11] that there is a long exact sequence,

$$\cdots \longrightarrow H^j_{U_1}(M) \longrightarrow H^j_{T_i}(M) \longrightarrow H^{j+1}_{U_{i+1}}(M) \longrightarrow H^{j+1}_{U_1}(M) \longrightarrow \cdots$$

As $H^j_{U_i}(M) \cong H^j_{I_i}(M)$ for all $1 \leq i \leq c-1$ and for all $j \geq 0$, it follows from the above long exact sequence that $cd(T_i, M) = i$. $\square$

3 Lower Bound For Cohomological Dimension

The main purpose of this section is to establish a lower bound for cohomological dimension and, in this regard, we prove the following theorem which gives a sufficient condition for an integer t to be a lower bound for $\operatorname{cd}(I, M)$.

Theorem 3.1. *Let R be a Noetherian ring, M an R-module (not necessarily finitely generated) and I an ideal of R with $\dim(R/I + \operatorname{Ann} M) = d$. Let $t \geq 0$ be an integer. If there exists an ideal J of R such that $\mathrm{H}^{d+t}_{I+J}(M) \neq 0$, then t is a lower bound for $\operatorname{cd}(I, M)$. Moreover, if $\operatorname{cd}(I, M) = t$, then*

$$\mathrm{H}^d_J(\mathrm{H}^t_I(M)) \cong \mathrm{H}^{d+t}_{I+J}(M)$$

and $\dim \operatorname{Supp}(\mathrm{H}^t_I(M)) = d$.

Proof. Consider the Grothendieck's spectral sequence

$$E_2^{p,q} = \mathrm{H}_J^p(\mathrm{H}_I^q(M)) \Longrightarrow \mathrm{H}_{I+J}^{p+q}(M)$$

and look at the stage $p+q=n$. Since $\mathrm{Supp}(\mathrm{H}_I^q(M)) \subseteq V(I) \cap \mathrm{Supp}(M) \subseteq V(I+\mathrm{Ann}\, M)$, $\dim \mathrm{Supp}(\mathrm{H}_I^q(M)) \leq d$ for all q. Therefore it follows from Grothendieck's vanishing theorem that for all $p > d$, $E_2^{p,d+t-p} = 0$. But then since from the hypothesis $\mathrm{H}_{I+J}^{d+t}(M)$ does not vanish, there is at least one $p \leq d$ such that

$$E_2^{p,d+t-p} = \mathrm{H}_J^p(\mathrm{H}_I^{d+t-p}(M)) \neq 0.$$

Hence $\mathrm{H}_I^{d+t-p}(M) \neq 0$ and so $\mathrm{cd}(I, M) \geq d+t-p \geq t$.
If, in particular, $\mathrm{cd}(I,M) = t$, then $E_2^{p,q} = 0$ for all $q > t$. Now from the subsequent stages of the spectral sequence

$$E_k^{d-k,t+k-1} \longrightarrow E_k^{d,t} \longrightarrow E_k^{d+k,t-k+1}$$

and the fact that $E_k^{d-k,t+k-1} = E_k^{d+k,t-k+1} = 0$ for all $k \geq 2$, we have $E_\infty^{d,t} = E_2^{d,t}$. Hence $\mathrm{H}_J^d(H_I^t(M)) \cong \mathrm{H}_{I+J}^{d+t}(M)$.
Since $\mathrm{H}_J^d(\mathrm{H}_I^t(M)) \neq 0$, it follows from Grothendieck's vanishing theorem that $\dim \mathrm{Supp}(\mathrm{H}_I^t(M)) \geq d$. On the other hand, since $\dim \mathrm{Supp}(\mathrm{H}_I^t(M)) \leq \dim(R/I + \mathrm{Ann}\, M) = d$, we conclude that $\dim \mathrm{Supp}(\mathrm{H}_I^t(M)) = d$. □

So far, for a finitely generated R- module M, the best known lower bound for $\mathrm{cd}(I,M)$ is $\mathrm{ht}_M(I) = \mathrm{ht}\, I(R/\mathrm{AnnM})$. As an immediate consequence of Theorem 3.1, we sharpen this bound to $\dim(M) - \dim(M/IM) \geq \mathrm{ht}_M(I)$.

Corollary 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, M a finitely generated R-module of dimension n and I an ideal of R such that $\dim(M/IM) = d$. Then $n-d$ is a lower bound for $c = \mathrm{cd}(I,M)$. Moreover, if $c = n-d$, then*

$$\mathrm{H}_\mathfrak{m}^d(\mathrm{H}_I^{n-d}(M)) \cong \mathrm{H}_\mathfrak{m}^n(M)$$

and $\dim \mathrm{Supp}(\mathrm{H}_I^{n-d}(M)) = d$.

Proof. This follows from Theorem 3.1 and the fact that $H_\mathfrak{m}^n(M) \neq 0$. □

For an ideal I of R, it is a well-known fact that $\mathrm{ht}(I) \leq \mathrm{cd}(I) \leq \mathrm{ara}(I)$, where $\mathrm{ara}(I)$ denotes the smallest number of elements of R required to generate I up to radical. If, in particular, $\mathrm{ara}(I) = \mathrm{cd}(I) = \mathrm{ht}(I)$, then I is called a *set-theoretic complete intersection ideal.* Determining set-theoretic

complete intersection ideals is a classical and long-standing problems in commutative algebra and algebraic geometry. Many questions related to an ideal I to being a set-theoretic complete intersection are still open, see [12] for more details. Varbaro in [13] show that under certain conditions there exists ideals I satisfying the property that $\operatorname{cd}(I) = \operatorname{ht}(I)$, knowing the existence of ideals with such properties, we have the following:

Corollary 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension n and I an ideal of R with $d = dim(R/I)$ such that $\operatorname{cd}(I) = \operatorname{ht}(I) = h$. Then $\dim(R) = \operatorname{ht}(I) + \dim(R/I)$ and*

$$\mathrm{H}_{\mathfrak{m}}^{n-h}(\mathrm{H}_I^h(R)) \cong \mathrm{H}_{\mathfrak{m}}^n(R).$$

Proof. It follows from Corollary 3.2 that $\dim(R) - \dim(R/I) \leq \operatorname{cd}(I) = \operatorname{ht}(I)$, while the other side of the inequality always holds. Therefore $\dim(R) = \operatorname{ht}(I) + \dim(R/I)$. Now the required isomorphim follows from Corollary 3.2. □

We end this section with the following conclusion :

Corollary 3.4. *Let $(R, \mathfrak{m})$ be a non-catenary Noetherian local domain of dimension n. Then there is at least one prime ideal of R that is not a set theoretic complete intersection.*

Proof. Since R is non-catenary, there is a prime ideal $\mathfrak{p}$ of R such that $\operatorname{ht}(\mathfrak{p}) < n - \dim(R/\mathfrak{p})$. Then it follows from Corollary 3.3 that $\operatorname{cd}(\mathfrak{p}) \neq \operatorname{ht}(\mathfrak{p})$ and therefore $\mathfrak{p}$ can not be a set theoretic complete intersection ideal. □

4 Artinianness and Non-Artinianness of Top Local Cohomology Modules

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M an R-module of dimension n. Recall that if M is a coatomic or a weakly finite (in particular, finitely generated, I-cofinite, or a balanced big Cohen Macaulay) module, then $\mathrm{H}_{\mathfrak{m}}^n(M)$ is nonzero and Artinian [[2], [3]].
In light of this information, we have the following results the first of which is the generalization of Theorem 7.1.6 of [4]:

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M an R-module of dimension n such that $\mathrm{H}^n_{\mathfrak{m}}(M)$ is Artinian. Then $\mathrm{H}^n_I(M)$ is Artinian for all ideals I of R.*

Proof. We use induction on $d = \dim(R/I)$. If $d = 0$, then I is an $\mathfrak{m}$-primary ideal and so $\mathrm{H}^n_I(M) \cong \mathrm{H}^n_{\mathfrak{m}}(M)$ is Artinian.
Let now $d > 0$ and suppose the hypothesis is true for all ideals J of R with $\dim(R/J) < d$.
Choose an element $x \in \mathfrak{m}$ such that $\dim(R/(I + Rx)) = d - 1 < d$. Then by induction hypothesis, $\mathrm{H}^n_{I+Rx}(M)$ is Artinian. Now consider the exact sequence

$$\cdots \longrightarrow \mathrm{H}^n_{I+Rx}(M) \longrightarrow \mathrm{H}^n_I(M) \longrightarrow \mathrm{H}^n_I(M_x) \longrightarrow \cdots$$

Since $\mathrm{H}^n_{I+Rx}(M)$ is Artinian and $\mathrm{H}^n_I(M_x) = 0$ ($\dim(M_x) < n$, as $x \in \mathfrak{m}$), it follows from the above exact sequence that $\mathrm{H}^n_I(M)$ is Artinian. □

Recall that a class $\mathcal{S}$ of R-modules is a Serre subcategory of the category of R-modules, $\mathcal{C}(R)$, when it is closed under taking submodules, quotients and extensions. The main result of this section is the following:

Theorem 4.2. *Let R be a Noetherian ring, M an R-module (not necessarily finitely generated) and let $\mathcal{S}$ be a Serre subcategory of $\mathcal{C}(R)$. Let I and J be two ideals of R such that $\mathrm{H}^{t+i}_J(\mathrm{H}^{c-i}_I(M)) \in \mathcal{S}$ for all $0 < i \leq c = \mathrm{cd}(I, M)$ and $\mathrm{H}^{t+c}_{I+J}(M) \notin \mathcal{S}$ for some positive integer t. Then $\mathrm{H}^t_J(\mathrm{H}^c_I(M)) \notin \mathcal{S}$.*

Proof. Consider the Grothendieck's spectral sequence

$$E^{p,q}_2 = \mathrm{H}^p_J(\mathrm{H}^q_I(M)) \Longrightarrow \mathrm{H}^{p+q}_{I+J}(M)$$

and look at the stage $p + q = c + t$. Let now $0 < i \leq c = \mathrm{cd}(I, M)$. Since $E^{t+i,c-i}_\infty = E^{t+i,c-i}_r$ for sufficiently large r and $E^{t+i,c-i}_r$ is a subquotient of $E^{t+i,c-i}_2 \in \mathcal{S}$, $E^{t+i,c-i}_\infty \in \mathcal{S}$ for all $0 < i \leq c = \mathrm{cd}(I, M)$.
On the other hand, since $E^{t,c}_2 = \mathrm{H}^t_J(\mathrm{H}^c_I(M)) \Longrightarrow \mathrm{H}^{t+c}_{I+J}(M)$, there exists a finite filtration

$$0 = \Phi^{t+c+1}\,\mathrm{H}^{t+c} \subseteq \Phi^{t+c}\,\mathrm{H}^{t+c} \subseteq \cdots \subseteq \Phi^1\,\mathrm{H}^{t+c} \subseteq \Phi^0\,\mathrm{H}^{t+c} = \mathrm{H}^{t+c}$$

of $\mathrm{H}^{t+c} = \mathrm{H}^{t+c}_{I+J}(M)$ such that $E^{p,q}_\infty = \Phi^p\,\mathrm{H}^{t+c}/\Phi^{p+1}\,\mathrm{H}^{t+c}$ for all $p + q = t + c$. Since for all $p < t$, $E^{p,q}_\infty = 0$, we have that $\Phi^t\,\mathrm{H}^{t+c} = \cdots = \Phi^1\,\mathrm{H}^{t+c} =$

$\Phi^0 \operatorname{H}^{t+c} = \operatorname{H}^{t+c}$. But then since $E_\infty^{t+i,c-i} = \Phi^{t+i} \operatorname{H}^{t+c}/\Phi^{t+i+1} \operatorname{H}^{t+c} \in \mathcal{S}$ for all $0 < i \leq c$, $\Phi^{t+1} \operatorname{H}^{t+c} \in \mathcal{S}$ and so it follows from the short exact sequence

$$0 \longrightarrow \underbrace{\Phi^{t+1} \operatorname{H}^{t+c}}_{\in \mathcal{S}} \longrightarrow \underbrace{\operatorname{H}^{t+c}_{I+J}(M)}_{\notin \mathcal{S}} \longrightarrow E_\infty^{t,c} \longrightarrow 0$$

that $E_\infty^{t,c} \notin \mathcal{S}$. Since $E_\infty^{t,c}$ is a subquotient of $E_2^{t,c}$ and $E_\infty^{t,c} \notin \mathcal{S}$, it follows that $E_2^{t,c} = \operatorname{H}^t_J(\operatorname{H}^c_I(M)) \notin \mathcal{S}$. □

Corollary 4.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, M an R-module of dimension n such that $\operatorname{H}^n_{\mathfrak{m}}(M) \neq 0$ with $\dim(R/I + AnnM) = d$. If $\operatorname{H}_I^{\operatorname{cd}(I,M)}(M)$ is Artinian, then either $\operatorname{cd}(I, M) = n$ or $\operatorname{H}^{n-i}_{\mathfrak{m}}(\operatorname{H}^i_I(M)) \neq 0$ for some $n - d \leq i < \operatorname{cd}(I, M)$.*

Proof. We prove the contrapositive of the statement. Let $\mathcal{S}$ be the category of zero module and suppose that $c = \operatorname{cd}(I, M) < n$ and $\operatorname{H}^{n-i}_{\mathfrak{m}}(\operatorname{H}^i_I(M)) = 0 \in \mathcal{S}$ for all $n - d \leq i < c$. But then since $\operatorname{H}^n_{\mathfrak{m}}(M) \notin \mathcal{S}$, it follows from Theorem 4.2 that $\operatorname{H}^{n-c}_{\mathfrak{m}}(\operatorname{H}^c_I(M)) \neq 0$. Hence $\dim \operatorname{Supp}(\operatorname{H}^c_I(M)) > 0$ and so $\operatorname{H}^c_I(M)$ is not Artinian. □

The following results determine the Artinianness and non-Artinianness of the top local cohomology module, $\operatorname{H}_I^{\operatorname{cd}(I,M)}(M)$, for the ideals of small dimension.

Theorem 4.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, M an R-module of dimension n such that $\operatorname{H}^n_{\mathfrak{m}}(M)$ is nonzero and Artinian and let I be an ideal of R such that $\dim(R/I + \operatorname{Ann} M) = 1$. Then $\operatorname{H}_I^{\operatorname{cd}(I,M)}(M)$ is Artinian if and only if $\operatorname{cd}(I, M) = n$.*

Proof. Since $\dim(R/I + \operatorname{Ann} M) = 1$ and $\operatorname{H}^n_{\mathfrak{m}}(M) \neq 0$, it follows from Theorem 3.1 that either $\operatorname{cd}(I, M) = n - 1$ or $\operatorname{cd}(I, M) = n$. If $\operatorname{cd}(I, M) = n$, then by Theorem 4.1, $\operatorname{H}^n_I(M)$ is Artinian. If, on the other hand, $\operatorname{cd}(I, M) = n-1$, then it follows from Theorem 3.1 that $\dim \operatorname{Supp}(\operatorname{H}^{n-1}_I(M)) = 1$ and so $\operatorname{H}^{n-1}_I(M)$ is non-Artinian. □

Theorem 4.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, M an R-module of dimension n such that $\operatorname{H}^n_{\mathfrak{m}}(M)$ is nonzero and Artinian and let I be an ideal of R with $\dim(R/I + \operatorname{Ann}(M)) = 2$. If $\operatorname{H}_I^{\operatorname{cd}(I,M)}(M)$ is Artinian, then either $\operatorname{cd}(I, M) = n$, or $\operatorname{cd}(I, M) = n - 1$ and $\operatorname{H}^2_{\mathfrak{m}}(\operatorname{H}^{n-2}_I(M)) \neq 0$.*

Proof. Since $\dim(R/I + \operatorname{Ann} M) = 2$ and $\mathrm{H}^n_{\mathfrak{m}}(M) \neq 0$, it follows from Theorem 3.1 that $n-2$ is a lower bound for $\operatorname{cd}(I, M)$. If $\operatorname{cd}(I, M) = n-2$, then again by Theorem 3.1, $\dim \operatorname{Supp}(\mathrm{H}^{n-2}_I(M)) = 2$ and so $\mathrm{H}^{\operatorname{cd}(I,M)}_I(M)$ is non-Artinian. If, on the other hand, $\operatorname{cd}(I, M) = n$, then from Theorem 4.1, $\mathrm{H}^{\operatorname{cd}(I,M)}_I(M)$ is Artinian. Finally, if $\operatorname{cd}(I, M) = n-1$ and $\mathrm{H}^{n-1}_I(M)$ is Artinian, then the result follows from Corollary 4.3. $\square$

References

[1] M. Aghapournahr and L. Melkersson, *Artinianness of local cohomology modules,* Ark. Mat., **52**(2014), 1-10.

[2] M. Aghapournahr and L. Melkersson, *Finiteness properties of minimax and coatomic local cohomology modules*, Arch. Math. **94** (2010) 519–528.

[3] A. Bagheri, *'A Non-Vanishing Theorem for Local Cohomology Modules'*, Bull. Malays. Math. Sci. Soc. (2) **37** (2014) 65–72.

[4] M. P. Brodmann and R. Y. Sharp, *Local Cohomology*, 2nd ed., Cambridge Studies in Advanced Mathematics, Vol. 136, Cambridge University Press, Cambridge, 2013. An algebraic introduction with geometric applications.

[5] M. T. Dibaei and A. Vahidi, *Artinian and Non-Artinian Local Cohomology Modules,* Canad. Math. Bull., **54**(2011), 619-629.

[6] K. Divaani-Aazar, R. Naghipour and M. Tousi , *Cohomological dimension of certain algebraic varieties* , Proc. Amer. Math. Soc, **130**(12) (2002), 3537 3544.

[7] A. Grothendieck (notes by R. Hartshorne), *'Local cohomology'* , Lect. Notes in Math. **41** Springer-Verlag, 1967.

[8] R. Hartshorne, *Cohomological dimension of algebraic varieties,* Annals of Math., **88**(1968), 403-450.

[9] C. Huneke, *Problems in local cohomology, Free Resolutions in Commutative Algebra and Algebraic Geometry Sundance 90,* Res. Notes in Math., Jones and Barlett, **2**(1992), 93-108.

[10] C. Huneke and G. Lyubeznik, *On the vanishing of local cohomology modules,* Inv. Math., **102**(1990), 73-93.

[11] G. Lyubeznik, *A partial survey of local cohomology, local cohomology and its applications*,Lect. Notes Pure Appl. Math.,**226** (2002), 121-154.

[12] G. Lyubeznik , *A survey of problems and results on the number of defining equations. Commutative Algebra*, Math. Sci. Res. Inst. Publ., **15** (1989), 375-390.

[13] M. Varbaro, *Cohomological and Projective Dimensions* , Compositio Mathematica, **149**(2013), 1203-1210.